1.11 Rational Expressions

Polynomials have an arithmetic that is similar to the arithmetic of ordinary integers. In both systems, you can add, subtract, multiply, and divide, and these operations obey the same basic rules. You also have long division in both systems. You can use these facts to show that you can factor anything (into primes for an integer and into irreducibles for a polynomial) in exactly one way.

There may be more than one way to carry out the factorization, but there will only be one final result.

Factoring polynomials is useful when you want to solve polynomial equations. Factoring integers is useful when you want to do arithmetic with fractions. That is, when you want to simplify a fraction and when you want to add two fractions.

For You to Do

1. Just for old-time's sake, simplify $\frac{25{,}725}{86{,}625}$.

2. Find the sum $\frac{18}{49} + \frac{5}{21}$.

Because polynomials have an arithmetic (the basic rules of algebra) that is so much like the arithmetic of $\mathbb{Z}$, you can do arithmetic with fractions of polynomials.

Example 1

Problem Simplify $\frac{x^2 - 5x + 6}{x^2 - 9}$.

Solution

$$\begin{aligned}\frac{x^2 - 5x + 6}{x^2 - 9} &= \frac{(x-3)(x-2)}{(x-3)(x+3)} \\ &= \frac{\cancel{(x-3)}(x-2)}{\cancel{(x-3)}(x+3)} \\ &= \frac{(x-2)}{(x+3)}\end{aligned}$$

Notice that this expression is equivalent to the original everywhere except at $x = 3$. At $x = 3$, the original rational expression was undefined, but this new expression is equal to $\frac{1}{6}$ at $x = 3$.

When you get your ear pierced, your ear remains the same everywhere except at one point.

For You to Do

3. Simplify $\frac{x^3 - 1}{x^2 - 1}$.

Example 2

Problem Write this sum as a single rational expression.

$$\frac{5x + 1}{x^2 - 1} + \frac{3}{x - 1} + \frac{2}{x + 1}$$

Solution Just as with integers, you need to find a common denominator. Since $x^2 - 1 = (x - 1)(x + 1)$, all three denominators are factors of $x^2 - 1$. You can use that expression as the denominator. Multiply each fraction by a form of 1 that makes its denominator $x^2 - 1$.

$$\begin{aligned}\frac{5x + 1}{x^2 - 1} + \frac{3}{x - 1} + \frac{2}{x + 1} &= \frac{5x + 1}{x^2 - 1} + \frac{3}{x - 1} \cdot \frac{(x + 1)}{(x + 1)} + \frac{2}{x + 1} \cdot \frac{(x - 1)}{(x - 1)} \\ &= \frac{5x + 1}{x^2 - 1} + \frac{3(x + 1)}{x^2 - 1} + \frac{2(x - 1)}{x^2 - 1} \\ &= \frac{(5x + 1) + 3(x + 1) + 2(x - 1)}{x^2 - 1} \\ &= \frac{10x + 2}{x^2 - 1} \\ &= \frac{2(5x + 1)}{x^2 - 1}\end{aligned}$$

Habits of Mind

Think about it more than one way. If you are thinking of the polynomials as expressions, you call fractions like these **rational expressions.** If you are thinking of the polynomials as functions, you call them rational functions.

For You to Do

4. Write this difference as a single rational expression.

$$\frac{10}{x + 2} - \frac{6}{x - 2}$$

Exercises *Practicing Habits of Mind*

Check Your Understanding

1. Simplify each rational expression.

a. $\dfrac{15x}{5x^2}$

b. $\dfrac{x^2 - y^2}{(x + y)}$

c. $\dfrac{x^2 - y^2}{(x + y)^2}$

d. $\dfrac{x^2 - 1}{x^4 - 1}$

e. $\dfrac{2x^2 + x - 6}{3x^2 + 4x - 4}$

f. $\dfrac{3 - x - 3x^4 + x^5}{3 - x - 3x^3 + x^4}$

2. Let $f(x) = \dfrac{2x^2 + x - 6}{3x^2 + 4x - 4}$.
Suppose you define $g(x)$ by the same fraction, except the fraction is in simplest form. How do the graphs of $f(x)$ and $g(x)$ compare?

3. Write each sum as a single rational expression.

a. $\dfrac{b}{b - a} + \dfrac{a}{a - b}$

b. $\dfrac{1}{(x - a)(a - b)} + \dfrac{1}{(x - b)(b - a)}$

4. Write each sum as a single rational expression.

a. $\dfrac{1 + 2x}{3x - 3} + \dfrac{5 - x}{x^2 - 5x + 4}$

b. $\dfrac{2}{x - 3} - \dfrac{2}{x + 3} - \dfrac{1}{x}$

c. $\dfrac{1}{(a - b)(b - c)} + \dfrac{1}{(b - c)(c - a)} + \dfrac{1}{(c - a)(a - b)}$

5. **Take It Further** Find numbers A and B such that

$$\frac{1}{(x - 1)(x - 3)} = \frac{A}{(x - 1)} + \frac{B}{(x - 3)}$$

On Your Own

6. Simplify each rational expression.

a. $\dfrac{x^2 - 1}{x - 1}$

b. $\dfrac{x^4 - 1}{x^2 - 1}$

c. $\dfrac{x^6 - 1}{x^3 - 1}$

d. $\dfrac{x^8 - 1}{x^4 - 1}$

e. $\dfrac{x^{10} - 1}{x^5 - 1}$